*Journal of Engineering Mathematics* **30:** 547–555, 1996. 


# Torsion of a Non-homogeneous Infinite Elastic Cylinder Slackened by a Circular Cut

HASSAN A.Z. HASSAN
*Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt*

Received 29 June 1994; accepted in revised form 6 October 1995

**Key words:** Torsion, Nonhomogeneity, Elastic cylinder, Circular cut.

**Abstract.** We consider the torsional deformation of a non-homogeneous infinite elastic cylinder slackened by an external circular cut. The shear modulus of the material of the cylinder is assumed to vary with the radial coordinate by a power law. It is assumed that the lateral surface of the cylinder as well as the surface of the cut are free of stress. The main object of this study is to establish the effect of the non-homogeneity on the stress intensity factor at the tip of the cut. The problem leads to a pair of dual series relations, the solution of which is governed by a Fredholm integral equation of the second kind with a symmetric kernel. This equation is solved numerically by reducing it to an algebraic system. It is concluded that for any degree of non-homogeneity and for $D$, the relative depth of the cut, greater than 0.6, the cylinder may be replaced by a half-space. However, as the non-homogeneity increases, $D$ decreases.

## 1. Introduction

The problem of torsion of cylinders slackened by external cuts has important practical applications in engineering. It has been treated – for a homogeneous medium – by several authors, among whom we cite Kudriavtsev and Parton [1], Suzuki *et al.* [2], Shibuya *et al.* [3], and Zlatin and Uflyand [4]. The mathematical method used in these papers involves the reduction of the basic equations and boundary conditions to dual series equations. Sneddon and Srivastav [10], Srivastav [11] and Sneddon [12] have given methods to reduce the problem of solving dual relations involving Fourier Bessel and Dini series expansions to a Fredholm integral equation. The last equation can be solved either by transformation to an infinite system of algebraic equations or numerically.

The present work is an extension of the abovementioned problem to the case when the shear modulus of the material has an inhomogeneity described by a power-law dependence on the radial distance. Such a generalization finds its application in solid mechanics. This type of shear modulus can be possible in accreted bodies like a metallic shaft fabricated by sintering different materials or a multilayered metallic or polymeric cylinder.

The paper has six sections: Section 1 is the introduction; Section 2 contains the problem formulation and basic field equations and boundary conditions; Section 3 is concerned with the reduction of the problem to dual series equations which, in turn, are reduced to a single Fredholm integral equation of the second kind. We make use of a method similar to that used in ref. [4] with some generalizations. In Section 4 we present expressions for some physical quantities of interest in terms of the auxiliary function that solves the integral equation. Section 5 is devoted to the solution of the Fredholm integral equation by transforming it to an infinite system of linear algebraic equations, which is solved numerically by iteration. In

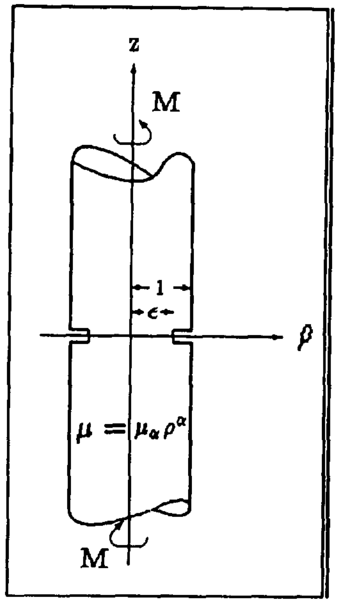


*Figure 1.* Geometry of the problem.

section 6, we present the numerical results and give conclusions concerning the influence of the inhomogeneity of the shear modulus on the stress intensity factor at the rim of the cut.

## 2. Formulation of the Problem

We consider a cylindrical rod of infinite length and unit radius slackened by an external circular cut. Introduce cylindrical coordinates $(\rho, \theta, z)$ with the axis of the crack as the $z$-axis. The crack is given by $z = 0, \epsilon \leq \rho \leq 1$. The lateral surface of the rod and the surface of the crack are assumed to be stress free. The rod is twisted by means of two equal and opposite torques $(M, -M)$ applied at infinity; see Figure 1. The shear modulus is taken in the form

$$\mu = \mu_\alpha \rho^\alpha, \tag{1}$$

where $\alpha \geq 0$; $\mu_\alpha$ is a constant.

For the torsional problems of bodies of revolution, the only non-vanishing component of the displacement vector is the $\theta$ component $v(\rho, z)$ which is independent of $\theta$. In the case of non-homogeneous and isotropic bodies this component must satisfy the equation [5]

$$\frac{\partial^2 v}{\partial \rho^2} + \left(\frac{1}{\rho} + \frac{\mu'_\rho}{\mu}\right)\frac{\partial v}{\partial \rho} - \left(\frac{1}{\rho} + \frac{\mu'_\rho}{\mu}\right)\frac{v}{\rho} + \frac{\partial^2 v}{\partial z^2} = 0, \tag{2}$$

where $\mu'_\rho = \partial\mu/\partial\rho$.

For the considered case, Equation (2) takes the form

$$\frac{\partial^2 v}{\partial \rho^2} + \frac{1+\alpha}{\rho}\frac{\partial v}{\partial \rho} - \frac{1+\alpha}{\rho^2}v + \frac{\partial^2 v}{\partial z^2} = 0. \tag{3}$$

The non-vanishing stress components are

$$\tau_{\theta_\rho}(\rho, z) = \mu\rho\frac{\partial}{\partial \rho}\left(\frac{v}{\rho}\right) \quad \text{and} \quad \tau_{\theta z}(\rho, z) = \mu\frac{\partial v}{\partial z}. \tag{4}$$

The symmetry of the problem with respect to the plane $z = 0$ makes it possible to search for a solution of (3) only in the semi-infinite cylinder $0 \leq \rho \leq 1, -\infty < z \leq 0$ with the boundary conditions

$$\tau_{\theta_\rho}(1, z) = 0, \qquad (-\infty < z \leq 0), \tag{5}$$

$$v(\rho, 0) = 0, \qquad (0 \leq \rho \leq \epsilon), \tag{6}$$

$$\tau_{\theta_z}(\rho, 0) = 0, \qquad (\epsilon < \rho \leq 1), \tag{7}$$

$$\int_0^1 \rho^2 \tau_{z\theta}(\rho, z)\, \mathrm{d}\rho = M/2\pi, \qquad (-\infty \leq z < 0). \tag{8}$$

The solution of Equation (3) satisfying conditions (5) and (8) is

$$v(\rho, z) = \frac{(4+\alpha)M}{2\mu_\alpha \pi} \rho z - C_0 \rho + \rho^{1-\nu} \sum_{k=1}^{\infty} C_k \lambda_k^{-1} J_\nu(\lambda_k \rho) e^{\lambda_k z}, \tag{9}$$

where $\nu = 1 + \alpha/2$, $\{\lambda_k\}$ are the positive zeros of the Bessel function $J_{\nu+1}(\lambda)$, and $\{C_k\}$ are constants to be determined.

The shear-stress component $\tau_{z\theta}(\rho, z)$ corresponding to the solution (9) is

$$\tau_{z\theta}(\rho, z) = \frac{(4+\alpha)M}{2\mu} \rho^{2\nu-1} + \mu_\alpha \rho^{\nu-1} \sum_{k=1}^{\infty} C_k J_\nu(\lambda_k \rho) e^{\lambda_k z}. \tag{10}$$

Thus, the remaining boundary conditions (6), (7) lead to the following system of dual series equations

$$-C_0 \rho + \rho^{1-\nu} \sum_{k=1}^{\infty} C_k \lambda_k^{-1} J_\nu(\lambda_k \rho) = 0, \qquad (0 \leq \rho \leq \epsilon), \tag{11}$$

$$\gamma \rho^\nu + \sum_{k=1}^{\infty} C_k J_\nu(\lambda_k \rho) = 0, \qquad (\epsilon < \rho \leq 1), \tag{12}$$

where

$$\gamma = \frac{4+\alpha}{2\pi\mu_\alpha} M = \frac{(1+\nu)M}{\pi\mu_\alpha}. \tag{13}$$

The shear stress acting on the plane $z = 0$ is

$$\tau_{z\theta}(\rho, 0) = \mu_\alpha \rho^{\nu-1} \left[ \gamma \rho^\nu + \sum_{k=1}^{\infty} C_k J_\nu(\lambda_k \rho) \right], \qquad (0 \leq \rho < \epsilon). \tag{14}$$

## 3. The Solution of the Dual Series Equations

To solve the pair of Equations (11) and (12), we make use of the method of Srivastav [11]. We extend Equation (12) to

$$\gamma \rho^\nu + \sum_{k=1}^{\infty} C_k J_\nu(\lambda_k \rho) = T(\rho), \qquad (0 \leq \rho < 1), \tag{15}$$

where

$$T(\rho) = \begin{cases} 0, & (\epsilon < \rho < 1), \\ -\rho^{\nu-1} \dfrac{\mathrm{d}}{\mathrm{d}\rho} \displaystyle\int_\rho^\epsilon \frac{g(t)}{\sqrt{t^2-\rho^2}}\,\mathrm{d}t, & (0 \leq \rho < \epsilon), \end{cases} \tag{16}$$

and the unknown auxiliary function $g(t)$ is assumed to be continuously differentiable in the closed interval $[0, \epsilon]$.

Then, on the basis of Equation (15) and the orthogonality of the system of functions $\{\rho^\nu, J_\nu(\lambda_k \rho)\}$ in the interval [0, 1] with the weight function $\rho$, we get

$$\gamma = 2(1+\nu) \int_0^1 \rho^{\nu+1} T(\rho)\,\mathrm{d}\rho = \frac{2\sqrt{\pi}\Gamma(\nu+2)}{\Gamma(\nu+1/2)} \int_0^\epsilon t^{2\nu-1} g(t)\,\mathrm{d}t, \tag{17}$$

$$C_k = \frac{2}{J_\nu^2(\lambda_k)} \int_0^1 \rho T(\rho) J_\nu(\lambda_k \rho)\,\mathrm{d}\rho, \qquad (k \geq 1). \tag{18}$$

Substitution from (18) into Equation (11) we get the following integral equation for determining the auxiliary function $g(t)$:

$$-C_0 \rho^\nu + \int_0^1 E(\rho, \eta) T(\eta) \eta\,\mathrm{d}\eta = 0, \qquad (0 \leq \rho \leq \epsilon), \tag{19}$$

where

$$E(\rho, \eta) = 2 \sum_{k=1}^{\infty} \frac{J_\nu(\lambda_k \eta) J_\nu(\lambda_k \rho)}{\lambda_k J_\nu^2(\lambda_k)}. \tag{20}$$

The kernel (20) can be expressed in integral form through the use of contour integration of the function of a complex variable

$$F(z) = \frac{H_{\nu+1}^{(1)}}{J_{\nu+1}(z)} J_\nu(\rho z) J_\nu(\eta z) + i \frac{4(1+\nu)\rho^\nu \eta^\nu}{\pi z^2},$$

where $H_\nu^{(1)}(z)$ is the Hankel function of the first kind and order $\nu$. The second term

$$i \frac{4(1+\nu)\rho^\nu \eta^\nu}{\pi z^2}$$

is taken to remove the singularity of the function $F(z)$ as $z$ as $z \to 0$. The integral is taken around the contour described in the paper of Sneddon and Srivastav [10]. One obtains the integral expression

$$\begin{aligned} E(\rho, \eta) \;=\; & \int_0^\infty J_\nu(\rho y) J_\nu(\eta y)\,\mathrm{d}y \\ & + \frac{2}{\pi} \int_0^\infty \left[ \frac{K_{\nu+1}(y)}{I_{\nu+1}(y)} I_\nu(\rho y) I_\nu(\eta y) - 2)\nu + 1) \frac{\rho^\nu \eta^\nu}{y^2} \right] \mathrm{d}y. \end{aligned} \tag{21}$$

Substituting (16) and (21) into (19) and using the Weber – Schafheitlin integral [6, form. 8.11(7)], and Sonine's first finite integral [13, form. 12.11(1)], we obtain a simple integral equation of Abel type:

$$\int_0^\rho \frac{t^{2\nu-1} g(t)}{\sqrt{\rho^2 - t^2}}\,\mathrm{d}t = \psi(\rho), \qquad (0 \leq \rho \leq \epsilon), \tag{22}$$

where

$$\psi(\rho) = C_0\rho^{2\nu} + \frac{2}{\sqrt{\pi}}\int_0^{\epsilon} g(t)\int_0^{\infty}\left[\frac{2\Gamma(\nu+2)}{\Gamma(\nu+1/2)}\rho^{2\nu}\frac{t^{2\nu-1}}{y^2} - \right.$$
$$\left. -\sqrt{\frac{y}{2}}\,t^{\nu-1/2}\frac{K_{\nu+1}(y)}{I_{\nu+1}(y)}\rho^{\nu} I_{\nu}(\rho y) I_{\nu-1/2}(ty)\right] \mathrm{d}y\,\mathrm{d}t. \tag{23}$$

Inverting the Abel Integral Equation (22), we obtain the following Fredholm integral equation of the second kind:

$$s^{\nu-1}g(s) = \frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+1/2)}C_0 s^{\nu} + \int_0^{\epsilon} K(s,t)t^{\nu-1}g(t)\,\mathrm{d}t, \tag{24}$$

with the symmetric kernel

$$K(s,t) = \frac{4}{\pi}\int_0^{\infty}\left[\frac{2\Gamma(\nu+1)\Gamma(\nu+2)}{[\Gamma(\nu+1/2)]^2}\frac{(ts)^{\nu}}{y^2}\right.$$
$$\left. -yt^{1/2}s^{1/2}\frac{K_{\nu+1}(y)}{2I_{\nu+1}(y)} I_{\nu-1/2}(ty)I_{\nu-1/2}(sy)\right]\mathrm{d}y. \tag{25}$$

Define

$$\frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+1/2)}C_0\phi(t) = t^{\nu-1}g(t). \tag{26}$$

It is seen that $\phi(t)$ satisfies the regular Fredholm integral equation of the second kind:

$$\phi(s) = s^{\nu} + \int_0^{\epsilon} K(s,t)\phi(t)\,\mathrm{d}t, \qquad (0 \le s \le \epsilon). \tag{27}$$

## 4. Some Physical Quantities of Interest

Now we express some quantities of physical importance in terms of the function $g(t)$. The shear component $\tau_{\theta_z}$ inside the region $z = 0$, $0 \le \rho \le \epsilon$ is found to be

$$\tau_{\theta_z}(\rho,0) = -\mu_{\alpha}\rho^{2\nu-2}\frac{\mathrm{d}}{\mathrm{d}\rho}\int_{\rho}^{\epsilon}\frac{g(t)\,\mathrm{d}t}{\sqrt{t^2-\rho^2}}, \qquad (0 \le \rho \le \epsilon). \tag{28}$$

The torque $M$ can be evaluated from Equations (13) and (17):

$$M = 2\mu_{\alpha}\pi^{3/2}\frac{\Gamma(\nu+1)}{\Gamma(\nu+1/2)}\int_0^{\epsilon} t^{2\nu-1}g(t)\,\mathrm{d}t. \tag{29}$$

The most interesting quantity for applications is the stress intensity factor

$$K_{III} = \lim_{\rho\to\epsilon}\sqrt{2\pi(\epsilon-\rho)}\,\tau_{\theta_z}(\rho,0) \tag{30}$$

which can be expressed in terms of the auxiliary function $g(t)$ by using Equations (28) and (29). We obtain

$$K_{III} = \frac{\Gamma(\nu+1/2)M}{2\pi\Gamma(\nu+1)}\frac{\epsilon^{2\nu-5/2}g(\epsilon)}{\int_0^{\epsilon} t^{2\nu-1}g(t)\,\mathrm{d}t}. \tag{31}$$

It is important to note that for small $\epsilon$ the integral of Equation (24) has the approximate solution

$$g(s) = \frac{2\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+1/2)} C_0 s.$$

This limiting case corresponds to the Reissner–Sagoci problem for a non-homogeneous half-space with shear modulus $\mu = \mu_\alpha \rho^\alpha$, studied in refs. [7, 8]. If we denote the stress intensity factor for this case by $K^\infty$, then

$$K^\infty = \frac{\Gamma(\nu+3/2)M}{\pi\Gamma(\nu+1)} \epsilon^{-5/2}.$$

We introduce the following dimensionless stress intensity factor:

$$K_p = \frac{K_{III}}{K^\infty} = \frac{\epsilon^{2\nu} g(\epsilon)}{(2\nu+1)\int_0^\epsilon t^{2\nu-1} g(t)\,\mathrm{d}t}. \tag{32}$$

## 5. Solution of the Integral Equation

Equation (24) or (27) with the kernel (25) can be solved numerically for general values of the parameters $\nu$ and $\epsilon$. Following Kantorovich and Krylov [9], the numerical solution of the integral equation can be obtained by replacing it by a finite system of linear algebraic equations. Zlatin and Uflyand [4] solved the homogeneous medium case by this method. On the other hand, the Integral Equation (27) can be reduced to an infinite system of linear equations as in [1]. The kernel (25) can be expressed in the form of a series

$$K(s,t) = -\frac{4}{\pi}(st)^\nu \sum_{m=0}^{\infty} b_m(t) s^{2m}, \tag{33}$$

where

$$\begin{aligned} b_m(t) &= \frac{1}{m!\Gamma(\nu+m+1/2)} \int_0^\infty \left\{ (y/2)^{\nu+2m+1/2} \frac{K_{\nu+1}(y)}{I_{\nu+1}(y)} t^{1/2-\nu} I_{\nu-1/2}(ty) \right. \\ &\qquad \left. - \frac{2\Gamma(\nu+1)\Gamma(\nu+2)}{y^2\Gamma(\nu+1/2)} \delta_{0m} \right\} \mathrm{d}y \\ &= \frac{1}{m!\Gamma(\nu+m+1/2)} \sum_{k=0}^{\infty} \frac{\Phi_\nu(m+k) t^{2k}}{k!\Gamma(\nu+k+1/2)}, \end{aligned} \tag{34}$$

$$\Phi_\nu(n) = \int_0^\infty \left[ \left(\frac{y}{2}\right)^{2(\nu+n)} \frac{K_{\nu+1}(y)}{I_{\nu+1}(y)} - \delta_{0n} \frac{2}{y^2} \Gamma(\nu+1)\Gamma(\nu+2) \right] \mathrm{d}y, \quad (n \geq 0) \tag{35}$$

and $\delta_{mn}$ is Kronecker's symbol.

We shall seek the solution of the integral equation in the form

$$\phi(s) = s^\nu \sum_{m=0}^{\infty} \gamma_m s^{2m}. \tag{36}$$

Substituting (33) and (36) in (27) we may show that the coefficients $\{\gamma_m\}$ are determined from the solution of the infinite system of algebraic equations

$$\gamma_m + \sum_{r=0}^{\infty} a_{mr}\gamma_r = \delta_{m0}, \quad (m \geq 0), \tag{37}$$

where

$$\begin{aligned} a_{mr} &= \frac{4}{\pi}\int_0^{\epsilon} t^{2\nu+2r} b_m(t)\,\mathrm{d}t \\ &= \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{\Phi_\nu(m+n)\epsilon^{2\nu+2n+2r+1}}{m!n!\Gamma(\nu+m+1/2)\Gamma(\nu+n+1/2)(2\nu+2n+2r+1)}. \end{aligned} \tag{38}$$

The system (37) is quasi-regular. In order to prove this, we shall first prove that

$$\lim_{m\to\infty} b_m(t) = 0, \qquad (0 \leq t < 1). \tag{39}$$

Substituting the asymptotic expansions for the modified Bessel functions in (34), we get the following leading term for large values of $m$:

$$b_m(t) \sim \frac{\sqrt{\pi}t^{-\nu}\Gamma(\nu+2m+1)}{m!\Gamma(\nu+m+1/2)\{2(2-t)\}^{\nu+2m+1}}.$$

This leads to (39).

Estimating the coefficients of the system (37) from (38), we get

$$|a_{mr}| \leq \frac{4}{\pi}|b_m(\epsilon')|\frac{\epsilon^{2\nu+2r+1}}{2\nu+2r+1}, \qquad (0 \leq \epsilon' \leq \epsilon < 1).$$

Consequently

$$\begin{aligned} S_m = \sum_{r=0}^{\infty}|a_{mr}| &\leq \frac{4}{\pi}|b_m(\epsilon')|\sum_{r=0}^{\infty}\frac{\epsilon^{2\nu+2r+1}}{2\nu+2r+1} \\ &< \frac{2}{\pi}|b_m(\epsilon')|\,|\ln(1-\epsilon^2)|. \end{aligned} \tag{40}$$

From the relations (39) and (40) it follows that the following inequality holds when $m$ starts from a certain number $m = N$:

$$S_m < 1, \qquad m \geq N.$$

Therefore, the system (37) is quasi-regular for $0 \leq \epsilon < 1$. The constant $N$ depends on $\epsilon$, it is larger, the closer $\epsilon$ is to unity. Thus we can solve the system (37) by truncation.

## 6. Numerical Results and Discussions

In the numerical examples the main question is the convergence behavior of the coefficients $\gamma_m$ and of the series giving the stress intensity factor $K_p$. Since the infinite system of simultaneous equations is to be truncated at a finite number, $N$, of terms, let Equation (37) be approximated by

$$\gamma_m + \sum_{r=0}^{N-1} a_{mr}\gamma_r = \delta_{m0}, \qquad (m = 0, 1, \ldots, N-1). \tag{41}$$

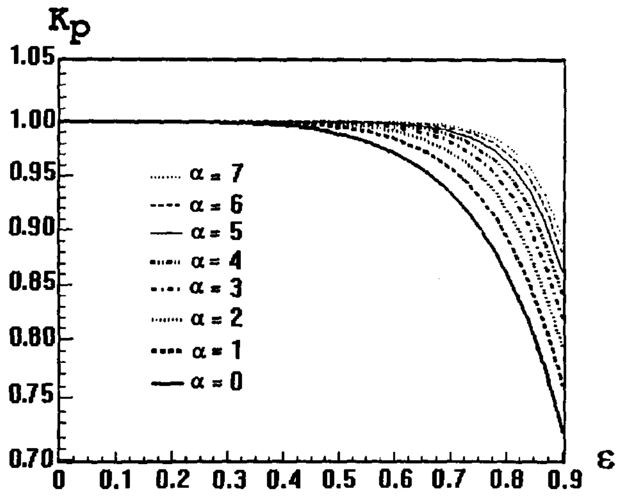


*Figure 2* The values of $K_p$ for $\alpha = 0, 1, \ldots, 7$.

On the basis of (26), (32) and (36), we have the following formula for calculating the nondimensional stress-intensity factor:

$$K_p = \left\{ \sum_{n=0}^{N-1} \gamma_n \epsilon^{2n} \right\} \bigg/ \left\{ (2\nu + 1) \sum_{n=0}^{N-1} \frac{\gamma_n}{2\nu + 2n + 1} \epsilon^{2n} \right\}. \tag{42}$$

The integrals (35) where evaluated numerically for the values $\alpha = 0, 1, \ldots, 7$ and $n = 0, 1, \ldots, 38$. The system (41) is solved. The stress intensity factor $K_p$ is calculated by the formula (42). The number $N$ is chosen to obtain a certain accuracy. The results are shown graphically in Figure 2, from which we conclude that:

(i) The stress intensity factor is almost the same as in the Reissner–Sagoci problem for all values of $\alpha$, provided that $\epsilon < 0.4$.

(ii) For $\epsilon > 0.4$, the stress intensity factor approaches $K_\infty$ as $\alpha$ increases.

## Acknowledgments

The referees are gratefully acknowledged for their careful reading and useful suggestions.

## References

1. B.A. Kudriavtsev and V.Z. Parton, Torsion and extension of a cylinder with an external annular slit, *PMM J. Appl. Math. Mech.* 37 (1973) 297–306.
2. K. Suzuki, T. Shibuya and T. Koizumi, The torsion of an infinite hollow cylinder with an external crack., *Int. J. Engng. Sci.* 16 (1978) 707–715.
3. T. Shibuya, T. Koizumi and T. Okuya, The axisymmetric stress field in an infinite solid cylinder with an external crack under torsion., *Bull. JSME.* 22 (1979) 1049–1052.
4. A.N. Zlatin and Ya. S. Uflyand, Torsion of an elastic cylinder slackened by an external circular notch. I. The case of an infinite cylinder., *Journal of Elasticity*, 13 (1983) 207–214.
5. H.A.Z. Hassan, Reissner–Sagoci problem for a non-homogeneous large thick plate, *J. de Mécanique* 18 (1979) 197–206.
6. A. Erdelyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Tables of Integral Transforms*, Vol. 2 New York: McGraw-Hill (1954).
7. B.M. Singh, A note on Reissner–Sagoci problem for a non-homogeneous solid, *ZAMM*, 53 (1973) 419–420.

8. H.A.Z. Hassan and F.E.K. Rofaeel, Reissner–Sagoci problem for a non-homogeneous half-space with surface constraint., *Journal of Engineering Mathematics* 28 (1994) 117–127.
9. L.V. Kantorovich and V.I. Krylov, *Approximate Methods of Higher Analysis*. Groningen The Netherlands: Noordhoff. (1964).
10. I.N. Sneddon and R.P. Srivastav, Dual series relations I, *Proc. Roy. Soc. Edinburgh A* 66 (1962–1963) 150–160.
11. R.P. Srivastav, Dual series relations II, *Proc. Roy. Soc. Edinburgh A* 66 (1962–1963) 161–171.
12. I.N. Sneddon, *Mixed Boundary Value Problem in Potential Theory*. Amsterdam: North-Holland Publ. Comp. (1966).
13. G.N. Watson, *A Treatise on the Theory of Bessel Functions*. Cambridge: Univ. Press (1966) pp. 805.